Frequency map analysis for beam halo formation in high intensity beams.

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Abstract

The Frequency Map analysis is applied to the Particle-core model of an intense bunched beam in a focusing channel with cylindrical symmetry. The coupled longitudinal-radial motion is analyzed.

1 INTRODUCTION

The control of beam losses down to a very small percentage is one of the main challenges for the new generation of high power linacs. These losses are associated with the presence of a beam halo, populated by very few particles but with a radius significantly larger than the beam rms (root mean square) radius up to the bore hole. In previous papers we have applied the Frequency Map Analysis (FMA) [1][2] to the 2D case of a mismatched beam propagating in a FODO channel[3]; in this case it exists a self-consistent solution of the Vlasov-Poisson problem, the KV distribution, and we studied how the non linear resonances, driven by the space charge, can push a test particle to large amplitudes. The importance of the envelope oscillations and the advantage of a careful choice of the working point result clearly from this analysis.

Unfortunately in a linac the particle dynamics is intrinsically 3D, since the three tunes are close each other. The space charge can be in this case estimated assuming an ellipsoid uniformly populated, even if this does not correspond to a regular self-consistent solution of Vlasov problem. In this paper we move a first step toward the 3D problem, by considering a beam moving in a solenoid channel, interrupted by an RF gap (working with -90 deg synchronous phase). The envelope periodic solution (Fig. 1) and the three envelope modes are calculated. This problem is very similar to a linac with smooth quadrupole focusing and same focusing strength in the two transverse planes; the understanding of the coupled longitudinal and transverse motion in this case is important for the halo formation problem. We therefore analyzed the 2D motion of particles without angular momentum respect to beam axis; in this way we could apply the FMA and the stability criteria elaborated for the transverse case.

The extension of the FMA to the complete 3D case will be our next step.

2 PARTICLE-CORE MODEL

We assume that the particles are uniformly distributed into ellipsoidal bunches, defined by the equation

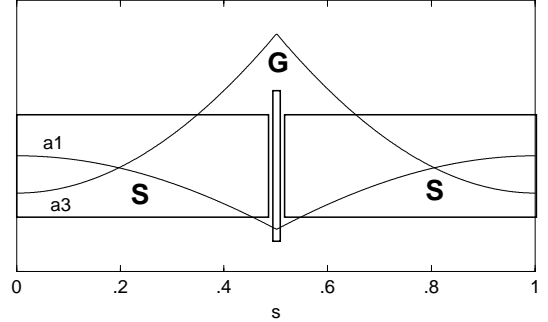$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \qquad (1)$$

Figure 1: FODO Period with $a_1 = a_2$

where (x_1, x_2) are the transverse coordinates and x_3 is the longitudinal coordinate. The semi axis a_i are functions of position s. We compute the potential of the charge distribution according to

$$\Phi = \frac{3Ne}{16\pi\epsilon_0} \int_\chi^\infty \frac{\left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1\right) du}{\sqrt{(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)}} \qquad (2)$$

where N is the total number of particle and the variable χ is set equal to 0 if the point (x_1, x_2, x_3) is internal to the ellipsoid (1), and is otherwise defined as the positive solution of the equation

$$\frac{x_1^2}{a_1^2 + \chi} + \frac{x_2^2}{a_2^2 + \chi} + \frac{x_3^2}{a_3^2 + \chi} = 1 \qquad (3)$$

In order to simplify the problem we assume a rotational symmetry ($a_1 = a_2$). In such a case the integral (2) can be computed using elementary functions and the transverse and the longitudinal electric fields determine the single particle equations: ($j = 1, 2$)

$$x_j'' + K_j x_j - \frac{\mu(1 - f(p))}{2(a_1^2 + \chi)\sqrt{a_3^2 + \chi}} x_j = 0$$

$$x_3'' + K_3 \frac{\beta\lambda}{2\pi} \sin\frac{2\pi}{\beta\lambda} x_3 - \frac{\mu f(p)}{(a_1^2 + \chi)\sqrt{a_3^2 + \chi}} x_3 = 0 \qquad (4)$$

where $\mu = 3Ne^2/4\pi\epsilon_0 m\gamma^3 v^2$, γ and β are relativistic factors, λ is the wavelength and the $'$ denotes the derivative with respect to s, $p = \sqrt{a_3^2 + \chi}/\sqrt{a_1^2 + \chi}$

$$f(p) = \frac{1}{p^2 - 1} - \frac{p \log(p + \sqrt{p^2 - 1})}{(p^2 - 1)^{3/2}} \qquad (5)$$

is the form factor defined for $p \in \mathbb{R}^+$ by extending the log and the square root functions to complex values. Since a_j depends on s, the equations (4) are time-dependent canonical equations. Therefore in order to integrate eqs. (4) we

have solved the envelope equations which follow from the rms envelope equations using the relations $\langle x_i^2 \rangle = a_i^2/5$

$$a_j'' + K_j(s)a_j - \frac{\mu(1-f)}{(a_1+a_2)a_3} - \frac{{\epsilon_j}^2}{a_j^3} = 0.$$

$$a_3'' + K_3(s)a_3 - \frac{\mu f}{a_1 a_2} - \frac{{\epsilon_3}^2}{a_3^3} = 0 \qquad (6)$$

where $\epsilon_i = 5\sqrt{\langle x_i^2 \rangle \langle {x_i'}^2 \rangle - \langle x_i x_i' \rangle^2}$ $i = 1,2,3$ are the emittances. If we choose the periodic solution of the envelope equations, then it is possible to introduce the Poincarè map for the system (4) (matched case); otherwise in the generic case $a_i(s)$ are only quasi-periodic and the Poincarè map cannot be used to plot the phase space of the test-particle.

3 ENVELOPE MODES

If the deviations δ_i with $i = 1,2,3$ from the periodic envelope are small, they can be calculated from the linearization of (6), giving rise to envelope modes that enter single particle dynamics. In particular if the focusing is smooth ($\nu_i \langle\langle 1/4\rangle$, one can directly calculate the equilibrium envelopes $a_i = \sqrt{\frac{\epsilon_i L}{2\pi \nu_i}}$, and the zero space charge tunes:

$$\nu_{0j} = \sqrt{\nu_j^2 + \frac{\mu}{4\pi^2} \frac{L^2(1-f)}{(a_1+a_2)a_j a_3}}. \qquad (7)$$

$$\nu_{03} = \sqrt{\nu_3^2 + \frac{\mu}{4\pi^2} \frac{L^2 f}{a_1 a_2 a_3}}. \qquad (8)$$

The envelope modes are solution of the system:

$$\delta_i'' + \sum_{l=1}^{3} H_{il}\delta_l = 0. \qquad (9)$$

where in the case $a_1 = a_2$, the matrix H_{il} has the form:
$A = H_{11} = H_{22} = \nu_{01}^2 + 3\nu_1^2 + \frac{\mu(1-f)}{8a_1^2 a_3} - \frac{\mu f'}{4a_1^3}$,

$B = H_{33} = \nu_{03}^2 + 3\nu_3^2 - \frac{\mu f'}{2a_3^3}$,

$C = H_{12} = H_{21} = \frac{\mu(1-f)}{8a_1^2 a_3} - \frac{\mu f'}{4a_1^3}$,

$D = H_{31} = H_{13} = H_{32} = H_{23} = \frac{\mu(1-f)}{2a_1 a_3^2} + \frac{\mu f'}{2a_1^2 a_3}$,

with $f' = df(p)/dp$. The eigen-frequencies are:

$$\alpha_0 = \sqrt{A - C} \qquad (10)$$

$$\alpha_\pm = \sqrt{\frac{A+B+C}{2} \pm \sqrt{\left(\frac{A-B+C}{2}\right)^2 + 2D^2}} \qquad (11)$$

and the corresponding eigen-vectors are:

$$\vec{\delta}_0 = (-1/\sqrt{2}, 1/\sqrt{2}, 0) \qquad (12)$$
$$\vec{\delta}_- = (-\sin\phi/\sqrt{2}, -\sin\phi/\sqrt{2}, \cos\phi) \qquad (13)$$
$$\vec{\delta}_+ = (\cos\phi/\sqrt{2}, \cos\phi/\sqrt{2}, \sin\phi) \qquad (14)$$

with

$$\phi = \frac{1}{2}\arctan\left(\frac{2\sqrt{2}D}{A-B+C}\right). \qquad (15)$$

This is the mode mixing angle. In particular, if radial and longitudinal focusing strength are equal, the mixing angle is $\pi/4$. On the contrary if the difference in focusing strength is large the mixing angle tends to zero. The lattices of practical interest are smooth enough so that the three modes calculated in smooth approximation can be recognized.

For numerical simulations we have chosen a lattice with $a_1 = a_2$ and with $\epsilon_1 = \epsilon_3 = 10^{-6}$ m, $K_1 = K_2 = 1.5$ m^{-2}, $K_3 = 0.9$ m^{-2}, $L = 3\beta\lambda = 1$ m, $\mu = 2 \cdot 10^{-9}$ m. In this case the frequencies in smooth approximation are: $\nu_{01} = 0.16, \nu_{03} = 0.16, \nu_1 = 0.13, \nu_3 = 0.12$ and the envelope modes are: $\alpha_0 = 0.27, \alpha_- = 0.26, \alpha_+ = 0.30$ with a mix angle of $\phi = 0.51$ rad.
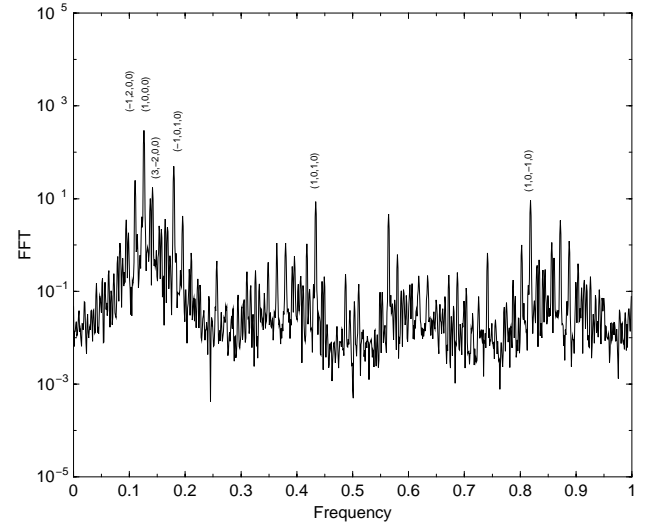

Figure 2: Fourier analysis of the mismatched case; (n_1, n_2, n_3, n_4) corresponds to $n_1\nu_1 + n_2\nu_3 + n_3\alpha_+ + n_4\alpha_- = 0$. Numerically $\nu_1 = .1247, \nu_3 = .1169, \alpha_+ = .3077, \alpha_- = .2657$

4 FREQUENCY-MAP ANALYSIS

The FM analysis is a useful tool to represent the phase space of a symplectic map of 2 or more degrees of freedom, which can be applied also to quasi-periodic time dependent maps. The basic idea is to compute the main frequencies associated to the invariant tori (regular orbits) and to study the regularity of the map (i.e. the FM) between the space of the invariant tori and the frequency space. According to the K.A.M. theory, in the regions of the phase space mostly filled by regular orbits the main frequencies associated to the invariant tori change smoothly from one torus to another, whereas in the resonant regions the frequencies are locked to the resonant values.
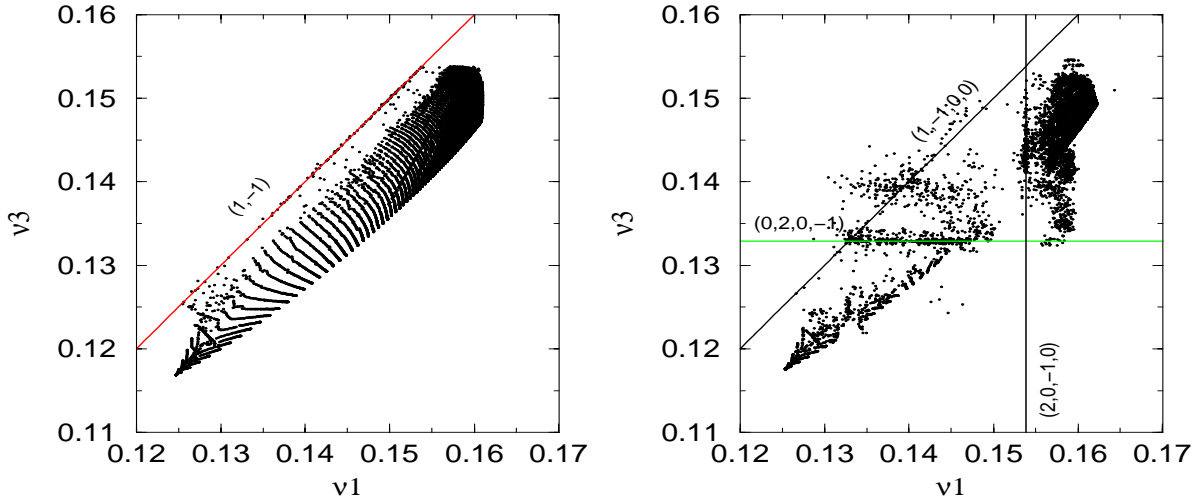
Figure 3: FMA in the matched and mismatched case: a $\delta_1 = 0.2a_1$ of mismatch is inserted in the envelope solutions: the low order resonances are indicated and the appearance of a chaotic region due to resonances overlapping for the mismatched case is clear.

In the case of the system (4), we numerically compute the transfer map for the FODO line and for a given initial condition we consider a fixed number of iterations. Then we project the orbit on the planes (x_i, x_i') $i = 1, 2, 3$ and we compute the betatronic and the synchrotronic frequencies by using an algorithm based on the Hanning filter.

In the figure 2 we show the FFT of a regular orbit for a test particle projected in a transverse plane when the beam envelopes are 20% mismatched. Even if the Fourier spectrum is very rich, due to the appearance of the integer combinations between the betatronic and synchrotronic frequencies and the envelope frequencies, we observe that the highest peak still corresponds to the betatronic frequency. This is in general true for the orbits of the system (4).

In order to study the regularity of the FM, we consider an uniform grid of initial conditions in the $(x_1 = x_2, x_3)$ plane (fig. 4). In the regular regions the initial grid of points is only smoothly deformed by the FM (fig. 3). In the resonant regions the the points are mapped in a resonant line so that a dense straight line of points at the center of an empty channel is shown in the frequency space. In the chaotic regions the results of the FM changes strongly from one point to another and the fuzzy cloud of points appears.

5 CONCLUSION

We have shown the possibility to use the FMA the longitudinal-radial coupled motion for high intensity bunched beam. The FMA provides information on the regular, resonant and chaotic regions in the phase space and allows to define a dynamic aperture by using the border of the regular regions around the beam core. In the mismatched case the FM analysis for the P-C model points out the role of the resonances between the single particle and the enve-

lope frequencies for the stability of the orbits, through the mechanism of resonance overlapping.
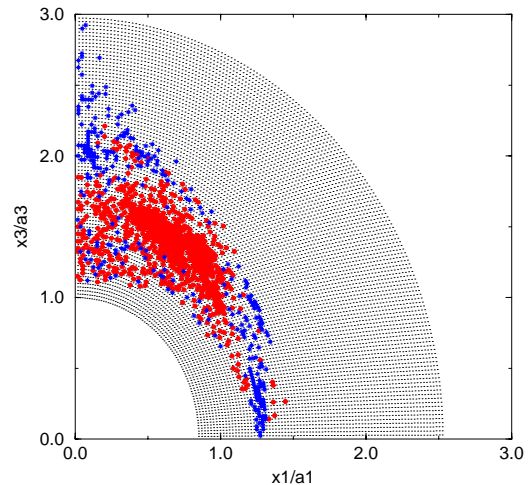

Figure 4: Initial condition corresponding to mismatched case (fig. 3 right side). The initial conditions corresponding to the two overlapping resonances $(2, 0, -1, 0)$ and $(0, 2, 0, -1)$ are indicated with thicker markers

6 REFERENCES

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