FREQUENCY MAP ANALYSIS OF THE CHAOTIC MOTION OF AN INTENSE MISMATCHED BEAM IN A FODO CHANNEL *

A.Pisent[1], A. Bazzani[2], Y. Papaphilippou[1,3], M. Comunian[1], G. Miano[4], V.G.Vaccaro[4], L.Verolino[4], G. Turchetti[2].

[1] *INFN-Laboratori Nazionali di Legnaro, Padova (Italy).*
[2] *INFN Sezione di Bologna and University of Bologna (Italy)*
[3] *Present address SL division, CERN (Switzerland)*
[4] *INFN Sezione di Napoli and University of Naples (Italy)*

Abstract

The comprehension of the mechanism that leads to small beam losses is one of the key points for the feasibility of the next generation of high power Linacs. We present here an analysis of the nonlinear dynamics of the beam halo particles in a FODO channel, based on the Frequency Map analysis. This tools allows the representation of the phase space also for the mismatched beam case. The influence of the image charges due to the presence of the vacuum pipe has also been considered.

1. Introduction

Proton Linacs with beam intensities between 10 and 120 mA, for a beam power up to 100 MW, are under study in various laboratories, for applications that go from the fundamental physics to energy production and nuclear wastes transmutation [1]. These performances will represent a big step forward respect to the present Linac technology, and one of the most critical aspects to allow this improvement is the control of beam losses. Typically losses lower than 1 W/m are needed to allow hand on maintenance in case of fault[2].

These losses are associated to the presence of a beam halo, populated by very few particles but with a radius of several times the beam rms (root mean square) radius up to the bore hole. A great theoretical effort is presently devoted to the understanding of halo formation [3] [4][5].

A realistic simulation of an intense beam in a Linac, able to follow 10^8-10^9 particles in

*Work partially supported by Human Capital and Mobility contract Nr. ERBCHRX-CT940480.

a self consistent way, taking into account the various scattering processes, interaction with the vacuum pipe, and whatever happens in a real Linac, is a too formidable task even for modern computers. Some simplifications are generally introduced, in order to determine more handy systems that include anyway the most relevant physical aspects. Many studies have been concentrated on the two degree of freedom problem of a collisionless continuous beam propagating in a FODO focusing channel. In this hypothesis it is easy to see in computer simulations, even with 1000-10000 particles, few particles getting rapidly to an amplitude 4-5 times the rms value (fig. 1). Bigger amplitude increases have been shown in more sophisticated simulations using a higher number of particles, especially in connection with mismatched envelope beams[6].

This behavior, due to the non-linear forces, is mainly determined by the space charge forces, acting on single particles. The problem can be simplified considerably considering the space charge forces as generated by the core of the beam (particle-core model), and calculating the core evolution using the method of the equivalent KV (Kapchinsky Vladimirsky) beam [7]. The single particle problem can be treated using the tools of dynamical systems. In particular particles immediately outside the core can reach big amplitudes and form the halo due to non-linear resonances and chaoticity in the phase space. The mechanism for the spill of few particles from the core to the halo could be for example a small non linearity of the space charge force inside the beam (deviation from KV distribution[5], image charges on the pipe..) or a low probability scattering process.

In this paper we have studied the beam dynamics in a FODO channel using the particle-core model. We have faced a specific problem: many simulations show that the halo formation is enhanced by the mismatching of the beam core. In this case the hamiltonian system associated to the betatron motion of the test particle is not periodically dependent on the longitudinal coordinate due to the non periodicity of the envelope of the beam. The direct plot of the phase space by using a Poincarè section does not allow to distinguish regular orbit from chaotic ones. In this paper we propose the method of the Frequency Map Analysis to represent the phase space [8]. This method has been applied in Celestial Mechanics [9] and Accelerator Physics [10,11,12] to study the stability of the orbits and turns out to be very efficient to detect the location of resonances and the chaotic regions. Moreover it is not affected by the non-periodicity of the Hamiltonian systems and can be extended to two or three degrees of freedom systems.

In section 2 we describe the particle in core model and we introduce the equation of motion for a test particle.

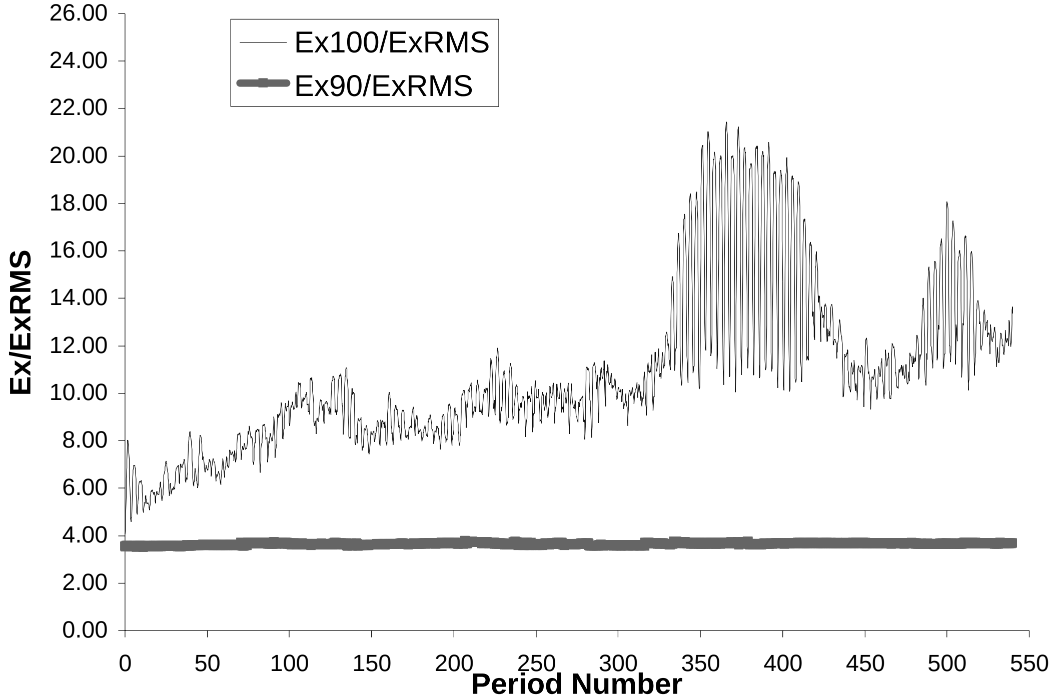
In section 3 we recall the effect of a mismatching on the envelope of the beam.

In section 4 we study the contribution of the fields induced by the beam on the vacuum pipe.

In section 5 we describe the method of the Frequency Map Analysis and we discuss its application to non periodic hamiltonian systems.

In section 6 we show the numerical results of our analysis on a 1 degree of freedom system describing a beam propagating in a FODO cell. The same method can be extended to the second degree of freedom following known procedures.

Figure 1



Multiparticle simulation (10,000 macroparticles, PARMILA code, standard PC version, LANL) of a long FODO channel, with the same parameters of the reference case studied in section 6 with particle core model. There is not rms emittance increase, and we plot here the ratios between the emittance containing 100% of particles and the rms value, and the same ratio for 90% of particles. The fluctuation of the 100% emittance is due to the irregular behavior of few particles.

2. The Particle-core model

The beam propagating in an accelerating structure can generally be treated as a Poisson-Vlasov problem: the particle distribution generates a field (self-field) that can be computed by solving the Poisson equation in the beam frame, and the distribution evolves according to a Vlasov equation, in which the superposition of external fields and self-fields is introduced. A solution of such a system is called self-consistent beam evolution. This approach is approximated, since collisions and effects of the complete electromagnetic system are neglected, but it is generally adequate for proton Linacs.

However when we look to very small beam losses, it is reasonable to assume that these losses are associated with the irregular behavior of few particles in the field generated by the regular particles, which form the "core" of the beam. This distinction between core and test particles is clearly a short cut, and gives solutions that are rigorously not self-consistent, but correct if irregular particles are few.

On the other hand this approach gives a single particle system and the possibility of an accurate analysis of the non-linear behavior of the particles, using all the tools offered by the Hamiltonian mechanics. If we introduce a set of coordinates $\vec{a}$ able to characterize the particle distribution, like the statistical moments (like $\overline{x}$, $\overline{x^2}$, $\overline{xy}$, $\overline{x^2y'}$, $\overline{xyy'}$..), the particle in core model can be written as:

$$\vec{a}(s) = \Phi(s, s_0, \vec{a}_0) \tag{2.1}$$

$$\vec{x}' = f(s, \vec{a}(s), \vec{x}) \tag{2.2}$$

where s is the longitudinal coordinate, $'$ indicates the derivative respect to s and $\vec{x}$ are the single particle coordinates; $\vec{a}_0$ refers to the initial position s_0. In the presence of a periodic focusing one is interested to the periodic distribution:

$$\vec{a}(s+L) = \vec{a}(s) \tag{2.3}$$

with L length of the period.

It is in principle possible to follow numerically the core of the beam, with the usual codes, and therefore to solve the single particle equation (2.2). In practice things are not so simple. The distribution is in general characterized by several statistical moments, so that the periodic solution (2.3) is very difficult to be found. Even to write explicitly the f function of equation (2.2) is difficult and it is in general easier to track the test particles in the actual field generated by the core particles.

Fortunately it exists a particular case in which a self-consistent periodic distribution for a beam propagating in a periodic focusing channel can be found: the KV distribution. A uniform charge density, with beam profiles corresponding to an ellipsis E of semi-axis a_x and a_y, can be characterized just by the second order momenta $\overline{x^2} = a_x/2$ and $\overline{y^2} = a_y/2$. The forces on the single particle can be calculated (by solving Poisson equation in elliptical coordinates) and correspond to the equations of motion:

$$\begin{aligned} x" + K(s)x - \frac{\xi}{(a_{x\chi} + a_{y\chi})a_{x\chi}} x = 0 \\ y" - K(s)y - \frac{\xi}{(a_{x\chi} + a_{y\chi})a_{y\chi}} y = 0 \end{aligned} \tag{2.4}$$

where the space charge parameter is

$$\xi = \frac{e}{\pi\epsilon_0} \frac{I}{mc^3\beta^3\gamma^3} = \frac{I}{I_c\beta^3\gamma^3},$$

I is the beam current (peak current for a bunched beam), $I_c = 7.8 \times 10^6 A$ is the critical current, β and γ are relativistic factors, $K(s)$ is the quadrupole external focusing. If $(x, y) \in E$ the parameters $a_{x\chi}$ and $a_{y\chi}$ are the semi-axis a_x and a_y of the ellipsis E, otherwise if $(x, y) \notin E$, $a_{x\chi}$ and $a_{y\chi}$ are the semi-axis of an ellipsis confocal to E where the point (x, y) located and are defined

$$a_{x\chi} = \sqrt{a_x^2 + \chi} \qquad a_{y\chi} = \sqrt{a_y^2 + \chi}$$

where χ is the solution of:

$$\frac{x^2}{a_x^2+\chi}+\frac{y^2}{a_y^2+\chi}=1.$$

The forces inside the beam are linear, and the equations of the envelope can be found with the substitution in (2.4) of the Floquet functions:

$$x=a_x(s)e^{i\psi_x(s)} \qquad y=a_y(s)e^{i\psi_y(s)}$$

with

$$\psi_x'=\frac{\epsilon_x}{a_x^2} \qquad \psi_y'=\frac{\epsilon_y}{a_y^2} \tag{2.5}$$

where the constants ϵ_x and ϵ_y are the emittances. The resulting envelope equations are:

$$\begin{aligned} a_x'' + K(s)a_x - \frac{\xi}{a_x+a_y} - \frac{{\epsilon_x}^2}{a_x^3} &= 0 \\ a_y'' - K(s)a_y - \frac{\xi}{a_x+a_y} - \frac{{\epsilon_y}^2}{a_y^3} &= 0. \end{aligned} \tag{2.6}$$

These equations, together with the single particle equations (2.4), give a coherent description of the dynamics and a self-consistent solution of the Poisson-Vlasov problem. The distribution that gives a homogeneous charge density and linear space charge forces is the KV distribution.

In accelerator physics the equations (2.6) are applied to more general case than the KV distribution. Indeed they are valid for any distribution if interpreted statistically, with $\overline{x^2}=a_x/2$ and $\epsilon_x=4\epsilon_{rmsx}$, with $\epsilon_{rmsx}^2=\overline{x^2}\,\overline{x'^2}-\overline{xx'}^2$, and analogous expressions for the vertical plane.

With distributions different from a KV, the space charge forces are not linear and the rms emittances are not constant. Nevertheless it has been shown that for many distributions of practical interest the emittance can be considered constant or as an adiabatic invariant, and the envelope equations (2.6) can be used as a good approximation of the rms behavior of the beam [13][14]. We shall adopt this point of view.

3. Periodic envelopes and envelope modes

In a periodic focusing channel one is interested to take the initial beam conditions in order to follow the periodic (matched) solution of the envelope equation:

$$\hat{a}_x(s+L)=\hat{a}_x(s) \qquad \hat{a}_y(s+L)=\hat{a}_y(s);$$

this allows the regular transport of the beam for an infinite (in principle) number of periods. The phase advance per period can be calculated from the equation (2.5) according to:

$$\sigma_x=\psi_x(s+L)-\psi_x(s)=\int_s^{s+L}\frac{\epsilon_x ds}{\hat{a}_x^2} \qquad \sigma_y=\psi_y(s+L)-\psi_y(s)=\int_s^{s+L}\frac{\epsilon_y ds}{\hat{a}_y^2};$$

for a pure quadrupole channel (focusing function opposite in vertical and horizontal plane) $\sigma_x = \sigma_y = \sigma$; the ratio σ/σ_0, with σ_0 phase advance corresponding to $\xi = 0$, called somehow improperly tune depression, gives a measure of the importance of the space charge in a specific case.

In real life the beam will be matched to the channel with an error:

$$a_x(s) = \hat{a}_x(s) + \delta_x(s) \qquad a_y(s) = \hat{a}_y(s) + \delta_y(s);$$

if this error is small the deviation from the periodic solution can be calculated from the linearized equations:

$$\delta_x" + \left[K(s) + \frac{\xi}{(\hat{a}_x + \hat{a}_y)^2} + \frac{3\epsilon_x^2}{\hat{a}_x^4}\right]\delta_x + \frac{\xi}{(\hat{a}_x + \hat{a}_y)^2}\delta_y = 0$$

$$\delta_y" + \left[-K(s) + \frac{\xi}{(\hat{a}_x + \hat{a}_y)^2} + \frac{3\epsilon_y^2}{\hat{a}_y^4}\right]\delta_y + \frac{\xi}{(\hat{a}_x + \hat{a}_y)^2}\delta_x = 0.$$

From the eigenmodes of the transfer matrix of a period, one can compute the frequencies of the envelope modes [15]. If $\sigma_0 << \pi$ the smooth approximation can be applied:

$$\hat{a}_x = \hat{a}_y = constant = a \qquad K = \left(\frac{\sigma_0}{L}\right)^2 \qquad \sigma = \frac{\epsilon L}{a^2} \qquad \sigma^2 - \sigma_0^2 = -\frac{\xi L^2}{2a^2}$$

and it is possible to decouple the two modes of oscillation:

$$(\delta_x + \delta_y)" + \frac{2(\sigma_0^2 + \sigma^2)}{L^2}(\delta_x + \delta_y) = 0$$

$$(\delta_x - \delta_y)" + \frac{\sigma_0^2 + 3\sigma^2}{L^2}(\delta_x - \delta_y) = 0$$

that are called respectively the even and the odd mode[15]; the lattices of practical interest are smooth enough so that those two modes can be recognized, and this simple estimate of their frequency ($\sigma_{even} = \sqrt{2(\sigma_0^2 + \sigma^2)}$ and $\sigma_{odd} = \sqrt{\sigma_0^2 + 3\sigma^2}$) is well fulfilled. If the beam is not matched the single particle equations (2.2) are not periodic, since:

$$\vec{a}(s + L) \neq \vec{a}(s)$$

and therefore it is impossible to use the Poincarè sections for the analysis of the orbits. We shall discuss in the section 5 a possible solution of this problem.

4. Effect of the vacuum pipe

The beam has in general an elliptical cross section in a circular vacuum pipe with radius b. The presence of the pipe, that restores the condition of cylindrical symmetry at radius b instead of infinity, induces additional fields that can be calculated with the method of the image charges. The term to be added to eq. (2.4) can be written, for a uniform charge density, in the closed form:

$$\frac{E_x + iE_y}{\gamma^3 \beta^2 mc^2} = \frac{\xi}{2} \frac{1}{x - iy} \frac{1 - \sqrt{1 - \frac{F^2}{b^4}(x - iy)^2}}{1 + \sqrt{1 - \frac{F^2}{b^4}(x - iy)^2}} \simeq \frac{\xi}{8} \frac{F^2}{b^4}(x - iy) + \frac{\xi}{16} \frac{F^4}{b^8}(x - iy)^3 + ..$$

with $F^2 = a_x^2 - a_y^2$. The linear part of this force enters into the envelope equation, with a (small) modification of the periodic solution and of the envelope modes. For example in smooth approximation the periodic (stationary) solution has circular cross section and is not affected by this field, as well as the even envelope mode; the odd mode frequency is instead depressed to:

$$\left[\frac{\sigma_{odd}}{\sigma_0}\right]^2 = 1 + 3\left[\frac{\sigma}{\sigma_0}\right]^2 - \frac{1}{2}\left[\frac{a}{b}\right]^4 \left\{1 - \left[\frac{\sigma}{\sigma_0}\right]^2\right\}.$$

For a completely space charge dominated beam ($\sigma = 0$) this mode is still stable, even if resonances can be met.

The non linear part of the force, present also inside the beam, could contribute to the spill of particles from the core to the halo. Anyway in the numerical cases we have simulated (one degree of freedom) we did not see any significative effect. The effect of the reactive part of the environment impedance, generated by the accelerating cavities, will be investigated later.

5. **Definition of the Frequency Map**

The idea to introduce the Frequency Map (F.M.) is due to J.Laskar [8,9] in order to represent the phase space of Hamiltonian systems with several degrees of freedom. For the sake of simplicity we shall discuss only the case of one degree of freedom (1D) Hamiltonian systems quasi-periodically dependent on time. More precisely we consider a time dependent Hamiltonian system of the form

$$H_0(x,p) + \epsilon H_1(x,p,\lambda_1,...,\lambda_m) \qquad \lambda_j = \omega_j t \qquad j \in [1,m] \tag{5.1}$$

where the dependence on the parameters λ_j is periodic of period 2π.

Let $x(t), p(t)$ any solution of the Hamiltonian system (5.1); for any interval Δt we consider the discrete orbit $(x_n, p_n) = x((n\Delta t), p(n\Delta t))$ and we compute the Discrete Fourier Transform (D.F.T.)

$$Z(\nu) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} (x_n + ip_n) e^{-2\pi i n \nu} \qquad \nu \in [0,1] \tag{5.2}$$

The F.M. associates to the orbit x_n, p_n the frequency corresponding the maximum value of $|Z(\nu)|$. If $\epsilon \ll 1$, by using the KAM theory it is possible to prove the existence of solutions which can be conjugated to an uniform translation on a $m+1$-dimensional torus with frequencies $\Omega_*, \omega_1, ..., \omega_m$. The KAM theorem requires the following conditions:

a) the frequencies $\Omega_*, \omega_1, ..., \omega_m$ are not resonant and satisfy a diophantine condition [16];
b) let $\Omega(I)$ the frequency of the unperturbed motion as a function of the action I then there exists a solution I_* to the equation $\Omega(I) = \Omega_*$ and $d\Omega/dI \gg \epsilon$.

In this case the orbit (x_n, p_n) can be expanded according to

$$x_n + ip_n = \sum_k a_k e^{ikn\Omega_* \Delta t} + \epsilon \sum_{k_1,...,k_m,k_{m+1}} b_k(\epsilon) e^{i(k_1\omega_1 + ... + k_m n\omega_m + k_{m+1}\Omega_*)n\Delta t} \tag{5.3}$$

so that the F.M. computes the frequency $k\Omega_*/2\pi$ corresponding to the maximal value of $|a_k|$. For most of the model which are relevant for Accelerator Physics, the unperturbed invariant curves are not too different from circles so that the maximum is achieved at $k = 1$ and the frequency $\Omega_*/2\pi$ is the nonlinear tune of the orbit.

If the orbit $x_n + ip_n$ lays on an invariant KAM torus the frequency $\Omega_*/2\pi$ characterizes the torus (and not the particular orbit); as a consequence the FM is correctly computed if we choose an initial condition for each invariant surface. If the frequencies are in an integer ratio with 2π, we define the Poincarè map with a period that is an integer combination of the fundamental periods, and we can choose the initial conditions on the Poincarè section. If the frequencies do not satisfy any resonant condition the Poincarè section can be defined as a limit process of the Poincarè maps associated with the rational approximations of the fundamental periods.

The behaviour of the F.M. is strictly related to the properties of the orbits: we distinguish three cases. If the orbits lie on a KAM tori then according to a theorem of Poeshel [17] there exists a $\mathcal{C}^\infty$ function which interpolates the frequencies $\Omega_*/2\pi$ and the F.M. is a regular monotonic curve.

In the case that the frequencies $\Omega_*, \omega_1, ..., \omega_m$ satisfy a resonant condition, by applying the Birkhoff theorem [18], we can prove the existence of a family of regular orbits whose frequency is locked to the resonance value (phase locking) so that the F.M. is constant. Moreover the derivative of the F.M. is divergent as we approach the external border of the region; as a consequence the F.M. detects the presence of very small resonances in the phase space.

Finally in case of chaotic orbits the Fourier spectrum is no longer discrete and for a fixed number of iterations N the result of the F.M. turns out to be very sensitive to the initial condition and we get an irregular curve.

The main property of the F.M. is the possibility of representing the phase space of the system (5.1) by using a single curve in a Poincarè section: in the case we are considering the x-axis will be a good choice. We consider a partition of the curve in equally space intervals; the length Δx of the intervals gives a measure of the scale at which we are analyzing the phase space. For each point of the partition we numerically compute the orbit up to a given iteration number N and then we evaluate the frequency corresponding to the maximum value of the D.F.T. $|Z(\nu)|$. The efficiency of the F.M. analysis is greatly improved by the existence of algorithms based on the Hanning filter which allows to compute the maximal value of $|Z(\nu)|$ with a precision $\propto 1/N^4$ if the distance of the frequencies $\Omega_*, \omega_1, ..\omega_m$ is $\gg 1/N$ [8],[11]; the complexity is equivalent to the FFT which usually gives a precision $\propto 1/N$.

For a 1D system the F.M. provides a curve as a function of the partition points. Of course the numerical smoothness of the curve depends on the ratio between precision of the tunes with the scale Δx which we use to explore the phase space; in order to get a smooth curve in a region of regular orbits we have to choose a sufficiently great iteration number N in order to reduce the ratio $\ll 1$; this condition has to be checked numerically.

The generalization to the 2D case is possible and the advantages of the F.M are even more relevant since it is possible to represent the phase space on a plane by using a surface in a Poincarè section.

6. Numerical Results

We have integrated the equations (2.4) and (2.5) for a FODO cell by means of a symplectic tracking program; the vertical amplitude of the test particle has been set to zero ($y = y' = 0$) so that the hamiltonian of the systems has the form (5.1). Both the quadrupole magnets and the empty parts have a length of $.2m$, as shown in figure 2. Various cases have been simulated; in the following of the paper we illustrate the method in the typical case:

$$\epsilon_x = \epsilon_y = 10^{-6} m., \qquad \xi = 4 \cdot 10^{-6}, \qquad K_{max} = 15.5 m^{-2}$$

correponding to

$$\frac{\sigma_0}{2\pi} = .1687, \qquad \frac{\sigma}{2\pi} = .084, \qquad \frac{\sigma_{odd}}{2\pi} = .21, \qquad \frac{\sigma_{even}}{2\pi} = .26,$$

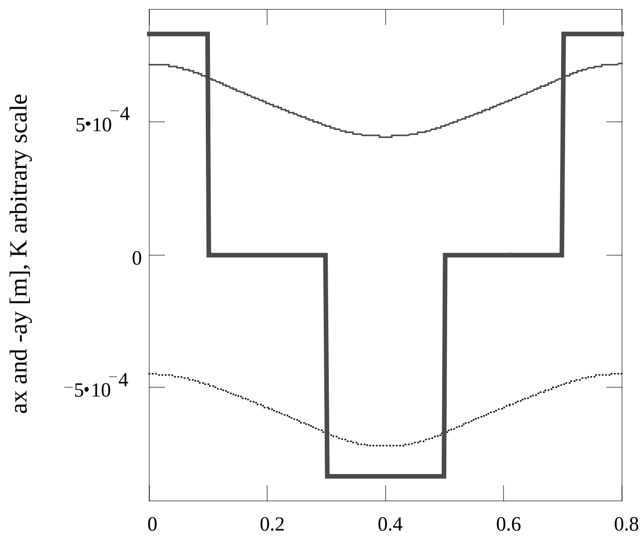
where σ_0 is calculated with the usual matrix composition, and the other frequencies in smooth approximation. The tracking program divides each element in 10 segments and the nonlinear potential due to the space charge is inserted by using a kick map in the center of each segment. We have checked the precision of the tracking by comparing with a Runge Kutta of order four. We have computed the initial conditions $\hat{a}_x, \hat{a}_y$for the periodic envelope and then we have excited (mainly) the odd mode by choosing the initial condition $\delta_x = 0.1\hat{a}_x, \delta_y = -0.1\hat{a}_y$.

In fig. 3 we plot the Fourier transform of an internal orbit for these cases called the matched and mismatched case. If the beam is matched we have a single peak corresponding to the betatron frequency $\nu = .0837555$; in the second case we observe various peaks which are due to the integer combination of three frequencies that can be recognized as the betatron frequency $\nu = .0845028$, the odd and the even envelope frequency ($\nu_{odd} = .2251448$ $\nu_{even} = .2673941$); the integer coefficients corresponding to each peak are reported on the figure. We observe that the betatronic frequency corresponds to the largest amplitude.

As a first step we show the results of F.M. analysis applied to the matched case. In fig. 4 we show the Poincarè section of the transverse phase space in the normalized coordinates

$$z_1 = \frac{x}{\hat{a}_x}, \qquad z_2 = \frac{x' \hat{a}_x}{\epsilon_x}$$

Figure 2



Geometry of the FODO; the focusing strength function K(s) and the matched envelopes are plotted

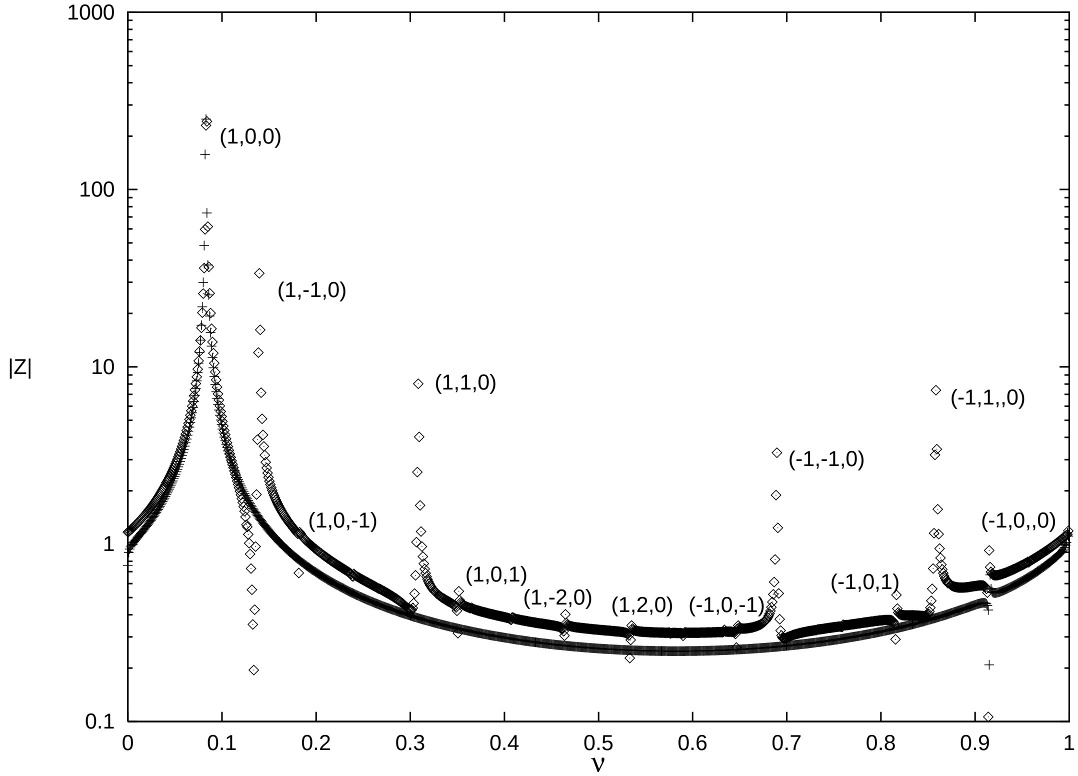
such that the matched beam fills the circle of unit radius.

In the top part of fig. 5 we show the F.M. analysis computed on a partition of 2,000 points on the x-axis (i.e. we use a scale .005 to explore the phase space); each orbit has been iterated $N = 10,000$ times and the estimated precision on the tune measurement is less than 10^{-6} for the regular orbits. The CPU time needed by the F.M. is half an hour on the Hp workstation 9000/755. The F.M. provided detailed information on the resonances (flat parts of the curve) in the phase space and we have reported on the curve the corresponding order of each resonances. The regular behavior of the curves means that the measure of the chaotic regions is small. This can also be observed in the bottom picture of fig. 5 where we report the maximal horizontal amplitude x_{max} of each orbit as a function of the initial condition. The regular increasing of x_{max} which denotes the absence of chaotic regions whereas the effect of resonances are the appearance of a valley.

In the case of a 10% mismatched beam, the Hamiltonian system of the FODO cell has the form (5.1) where $\omega_1/2\pi$ and $\omega_2/2\pi$ are the envelope frequencies of the of the beam. As a consequence the orbits in the phase space (z_1, z_2) become thick and it could be difficult to distinguish regular orbits from resonant or chaotic orbits directly from the picture; on the contrary the F.M. still works as in the matched case. The transverse phase space for the mismatched case is plotted in fig. 6; also in this case the coordinates are normalized on the matched envelopes.

The corresponding F.M. of the phase space is shown in fig.7 top; we have considered

Figure 3

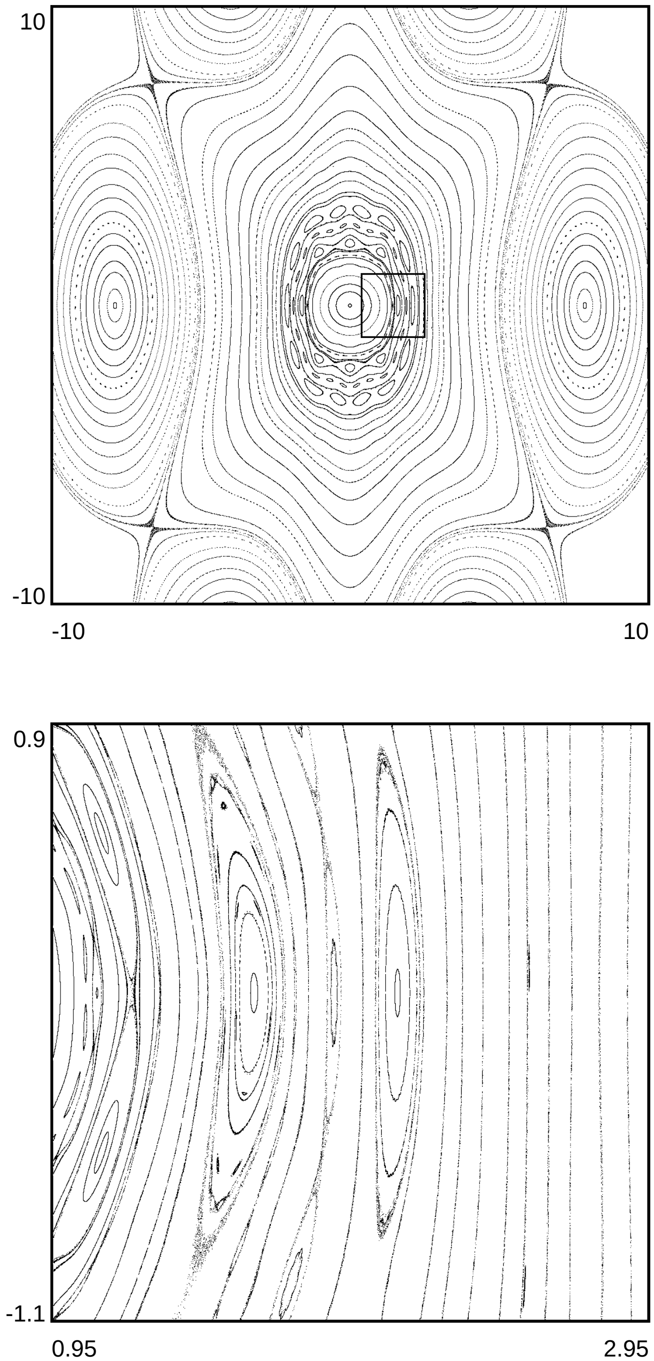


FFT of the orbit of particle inside the beam core; matched(crosses) and mismatched case(squares): in the last case nearby each peak we report the integer coefficients (k_1,k_2,k_3) which define the linear combination $(k_1\nu+k_2\nu_{odd}+k_3\nu_{even}$ of the betatronic frequency and the envelope frequencies corresponding to the peak. The initial conditions select the odd mode.

the same number of orbits and the same number of iterations than in the matched case. According to the theoretical results given in section 5 we can easily recognize the presence of resonances and chaotic regions from the irregular behavior of the curve. The initial flat part of the picture is due to the linear motion of the interior particles of the beam since we have assume the K.V.-distribution. On the pictures we have reported the integer coefficients $(k_1,k_2,k_3)$of each nonlinear resonance $k_1\nu + k_2\nu_{odd} + k_3\nu_{even} \in \mathbb{Z}$. In the last picture of fig. 7 we have reported the value of x_{max} for each orbit. We remark the correspondence between flat parts of this curve and the irregular behavior of the F.M. which means that there are chaotic orbits in the phase space that are ergodic in a bounded region. The chaotic region which appears just after the border of the beam, is due to the nonregularity of the space-charge field which is not differentiable at the border.

Our simple model could explain the presence of an halo up to an amplitude 2-3 times the dimension of the beam; indeed since the nonlinear space-charge potential is decreasing with the distance we recover a regular motion at larger amplitudes where a 6-order resonance is present. The effect of the unmatched initial condition for the envelopes is an enlargement of the chaotic region with respect to the matched case (compare with fig. 4).

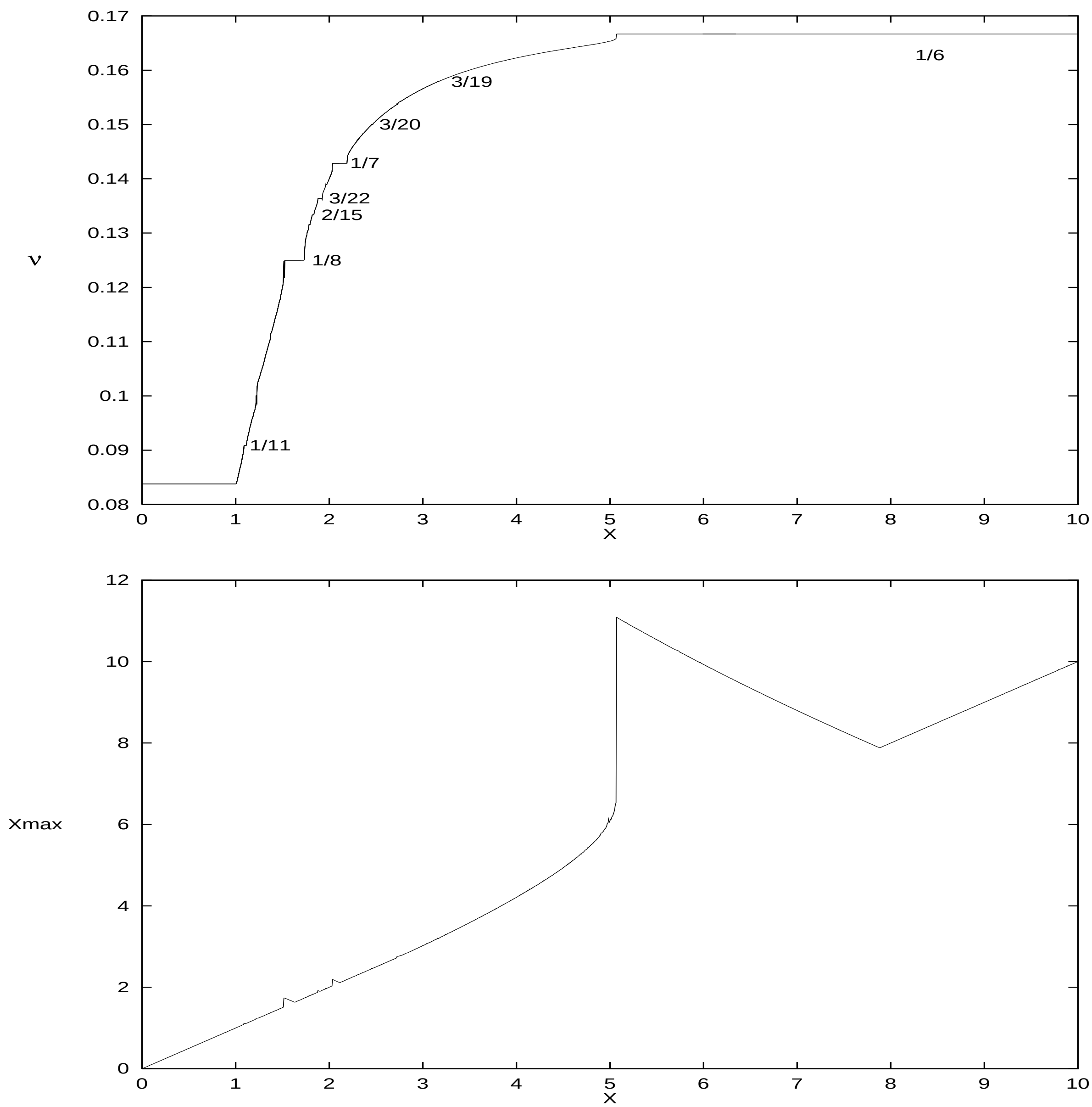
Figure 4



Phase-Space of a test particle in the matched case; the coordinates are scaled in such a way that the matched beam occupies the circle of unit radius.

Further studies are necessary to correlate the F.M. with the diffusion in the phase space which is more relevant for 2D systems. In order to illustrate the diffusion in the

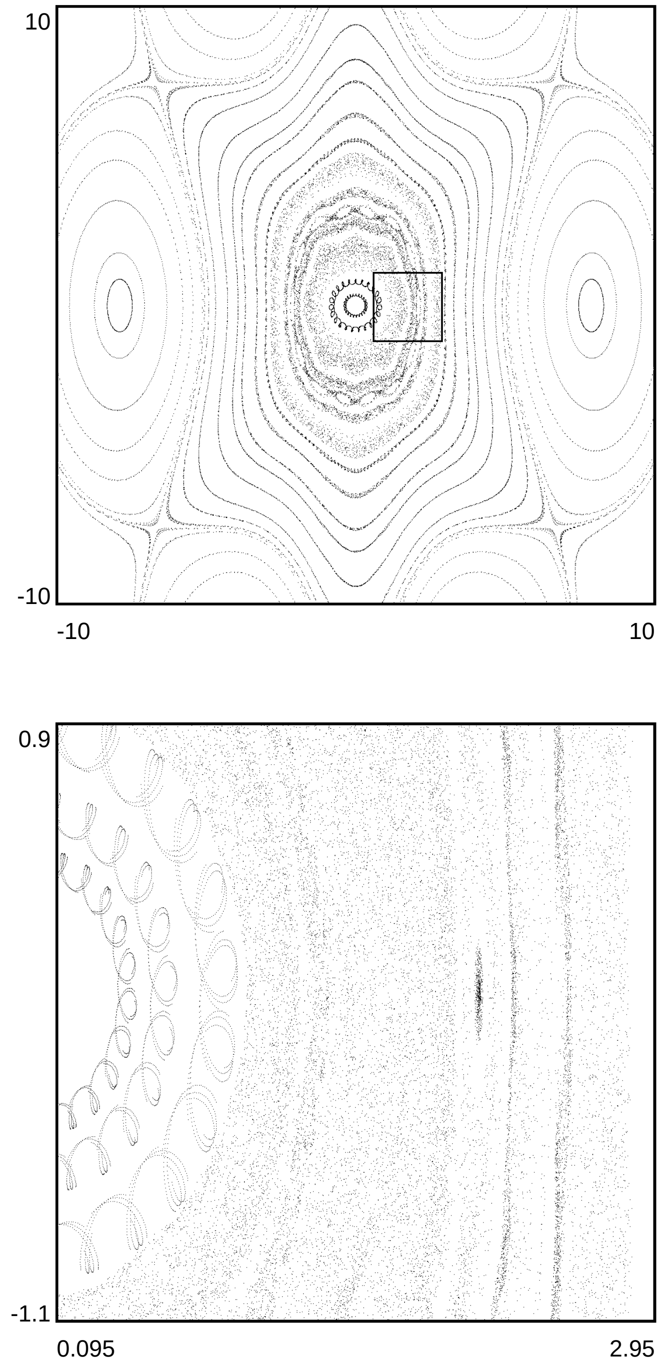
Figure 5



Top: F.M. analysis of the matched case by using 2000 points on the X-axis (in matched beam envelope units) as initial conditions: the frequency is computed by using 10,000 iterations for each orbits; we report on the picture the nonlinear resonances detected by the F.M.. Bottom: plot of the maximal horizontal amplitude as a function of the initial amplitude for the same orbit used in the F.M. analysis.

chaotic region for our model we report in the bottom part of fig. 5 the largest amplitude of each orbit as a function of the initial amplitude after 10,000 iterations. The flat parts of the picture corresponds to the chaotic regions where the orbits are ergodic, whereas a resonance in the phase space produce a valley due to the island structure. We remark the correspondence between the chaotic regions and the local behavior of the F.M.. If we

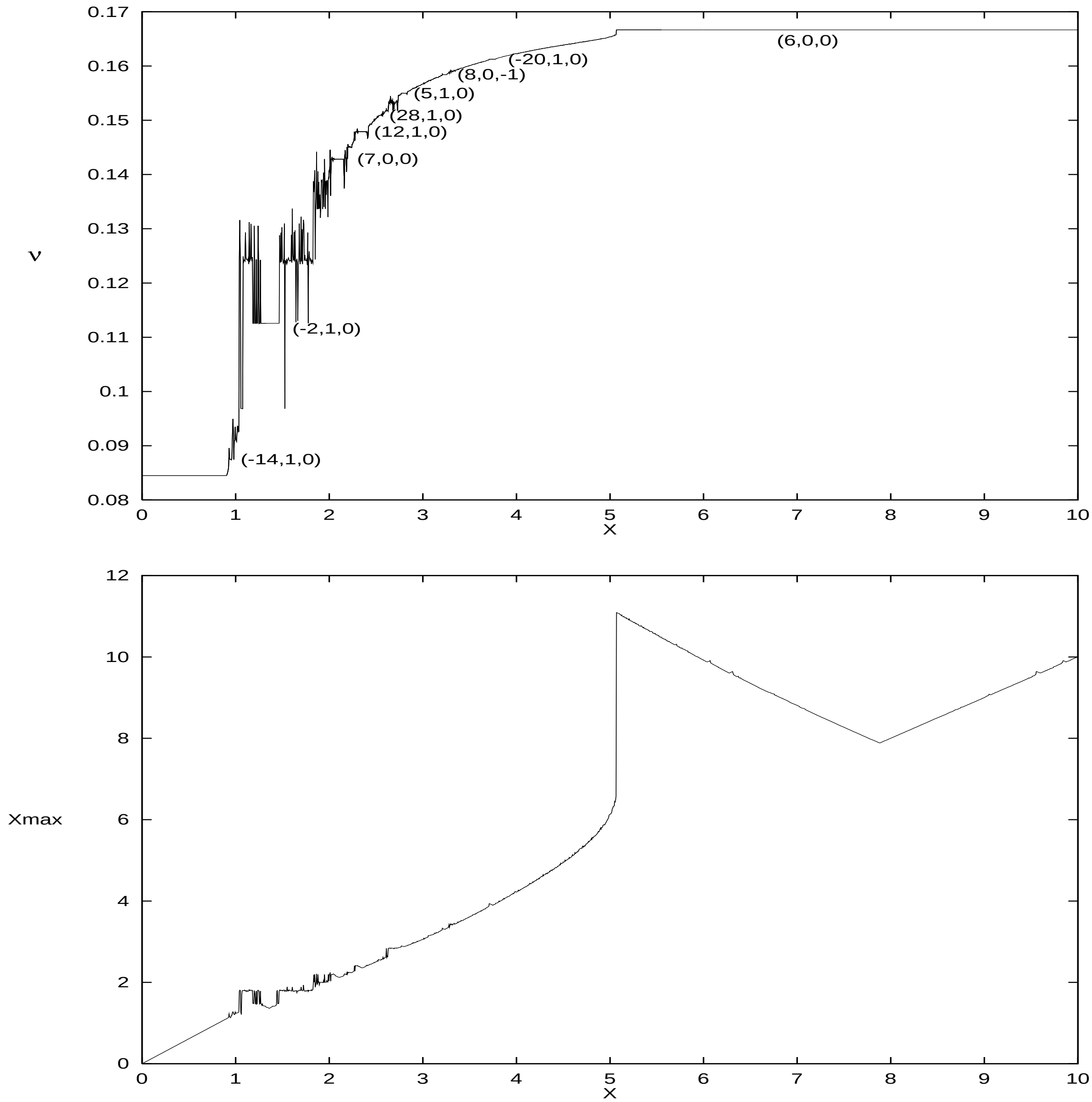
Figure 6



Phase-Space of a test particle in the mismatched case; the coordinates are scaled in such a way that the matched beam occupies the circle of unit radius.

have a small diffusion in the phase space the largest amplitude can substantially change as a function of the iterations number; on the contrary the F.M. picture is practically the

Figure 7



Top:F.M. analysis of the unmatched case by using 2000 points on the X-axis (in matched beam envelope units) as initial conditions: the frequency is computed by using 10,000 iterations for each orbits; we report on the picture the integer coefficients of the linear combination of the betatron frequency and the envelope frequencies which define the nonlinear resonances detected by the F.M.. Bottom: plot of the maximal horizontal amplitude as a function of the initial amplitude for the same orbit used in the F.M. analysis.

same. Then a possible strategy to study the diffusion in the phase space is to detect all the chaotic regions up to a certain scale and then to use a tracking program by considering only the orbits in these regions. This strategy could dramatically reduce the CPU time required for this analysis.

Conclusions

The F.M. turns out to be an efficient tool to represent the phase space of hamiltonian systems. It allows to detect the position of the resonances and the chaotic regions with an high accuracy and it can be used for time dependent systems with 1 or more degrees of freedoms when the direct plot of the phase space is not possible. Our analysis on a FODO cell in presence of a space-charge dominated beam suggests the possibility of using the F.M. to study the diffusion of the orbits in the phase space in the mismatched case.

References

[1] M. Prome, Proceedings of 1996 International Linac Conference, Geneva, CERN 96-07, p. 9 (1996).

[2] J.-M Lagniel, Proceedings EPAC 96 Conference, Sitges 1996, Institute of Physics Publishing Bristol and Philadelphia, (1996) p.210.

[3] R.L. Gluckstern, Proceedings of the 1994 international Linac Conference, Tsukuba, (1994) p. 333

[4] J.,-M Lagniel Nucl. Instr. Meth. , **A 345**, 405-410 (1994)

[5] Qian Qian, R.C. Davidson, C. Chen , Phys. Plasmas, **2(7)**, (July 1995) p. 2674

[6] A.6, M.Reiser, T. Wangler, Proceedings of 1991 PAC conference, IEEE 91CH3038-7 (1991) p.251

[7] I. Kapchinsky and V. Vladimirsky in Proccedings of the International Conference on High Energy Accelerators and Instrumentation, CERN 1959, p274.

[8] J.Laskar, Physica D, **67**, p. 257 (1993)

[9] J.Laskar, Icarus, **88** p. 266, (1990)

[10] H.S.Dumas, J.Laskar, Phys. Rev. Lett., **70**, p. 2975, (1993)

[11] R.Bartolini, A. Bazzani, M.Giovannozzi, W.Scandale, E.Todesco, Particle Accelerators **52**, p. 147, (1996)

[12] R.Bartolini, A.Bazzani, M.Giovannozzi, W.Scandale, E.Todesco, Particle Accelerators **55**, p. 247, (1996)

[13] F.J.Sacharer, IEEE Trans. Nucl. Sci. **NS-18**, 1105 (1971)

[14] P.Lapostolle, IEEE Trans. Nucl. Sci. **NS-18**, 1101 (1971)

[15] J.Struckmeier, N.Reiser, Part. Acc. **14**, p. 227-260 (1984)

[16] Arnold V.I., Russ. Math. Surv., **18**, n. 6, p. 85, (1963).

[17] Poeshel J., Commun. Pure Appl. Math. **35** n.1, p.653 (1982)

[18] Birkhoff G.D., American Mathematical Society, **VIII**, p. 295 (1927)